With bow and arrow

About the randomness of archery

Volker Jentsch
http://volkerjentsch.de

August 2023

One of the burning questions that archery enthusiasts[1] face is this: what is the chance that my arrow will hit the target? How can it be determined?

Consider the official $d_0 = 40\,cm$ target of the world archery. This contains eleven concentric rings and a cross in the centre (see Fig. 2). I assume that archery consists of a systematic and a probabilistic part. The systematic one is skill – experienced archers hit with 100% probability a target of $d_m = 10\,cm$ diameter from a distance of about $20\,m$. The random part is based on luck – the exact position of the hit within this circle is determined by uncontrollable factors, such as unintentional, small fluctuations in the archer's posture and aim. This part of the shot is thus interpreted as a random experiment whose results are equally probable and independent of each other.

The question to be answered is thus: How can I calculate the probability that the arrow hits a circle of diameter $d = d_m$, $d_m < d_0$. And here is the answer: With the help of geometric probability. This does not compare the number of favourable events with the number of possible events, as suggested by Laplace, but compares the favourable *area* (or *volume*) with the possible *area* (or *volume*).[2] The recipe for this is analogous to Laplace's concept of probability: divide the small circular area (the desired event) by the large circular area (the possible event)

$$p = \pi(d_m/2)^2/\pi(d_0/2)^2 \tag{1}$$

This (surprisingly) simple result is to be tested in a conventional way by repeated sequences of arrow-shooting. Let an arrow flow N times. A fraction n will arrive in the inner circle ($d_m = 8\,cm$), while $N - n$ hit the outer ring ($d_0 = 32\,cm$). Numbers N and n are obtained by means of a random number generator (which generates numbers from 0 to 1, ideally evenly distributed). According to (1), $p = 1/16$. This is the straight line in Fig.1. The fluctuating curve is $p^* = n/N$. Nice to see that p^* approaches p with increasing N, with amplitude

[1] I use a blank bow named *Black Wolf* from W&W with 36-40 lbs strength.

[2] I came across this consideration, which was unknown to me until now, with the help of the remarkable website *www.lerninhalten.de schülerlexikon.*

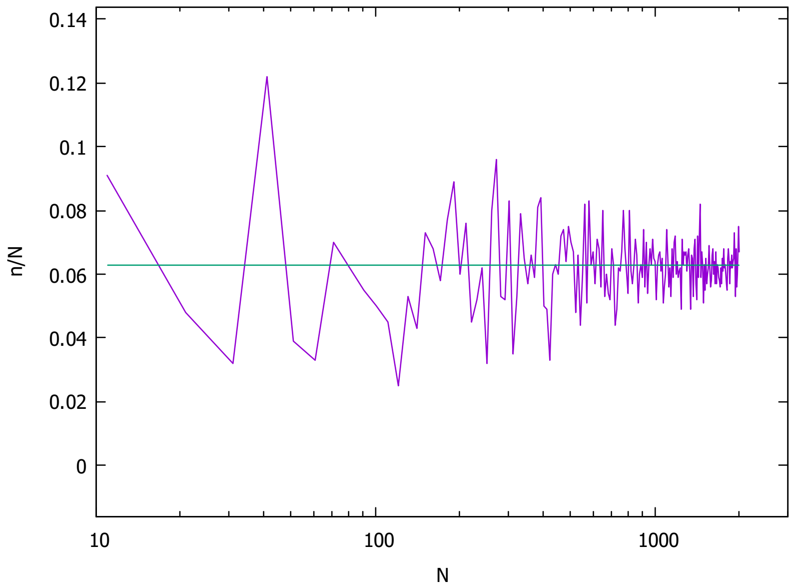

Figure 1: ratio of hits n in a sequence of N trials

$D \approx \pm 1/2\sqrt{N}$. Repeating the experiment would produce a different curve; which nevertheless, should converge towards the limit value p with increasing N. The advantage of (1) is obvious - the geometric probability saves thousends of shots! Incidentally, there is available an animation of fig.1 – visit `http://volkerjentsch.de/Bogen_1.html` .

Now, in possession of p, it is possible to solve the problem which is known as „Waiting for the first hit". Let P the probability, that the first hit occurs at the $k+1$ attempt. Therefore

$$P(k) = p(1-p)^k \tag{2}$$

It follows from (2) that with increasing number of attempts, the probability of scoring a hit decreases. However, more relevant in practice is the expectation value E. It gives the number of attempts that are required on average, i.e. in the long run, until the first hit is made in the $(k+1)$ attempt: and .

$$E(k) = (1-p)/p \tag{3}$$

with standard deviation σ:

$$\sigma(k) = \pm\sqrt{(1-p)/p^2} \tag{4}$$

Consequently, expectation and standard deviation increase as the probability of a hit decreases. For small values of p, $|\sigma| \approx E$, which limits its significance for practical purposes.

In the following, I discuss some applications.

Example 1: Archer Helmut is quite a skilled archer; he succeeds with certainty in hitting the target with a diameter of $d_0 = 32\,cm$, being $20\,m$ away from it. This diameter corresponds to the outer black ring of figure 2. For the inner ring, he chooses the maximum width of the yellow area; this is about $d_m = 8\,cm$ on the $40\,cm$ disc. Thus, according to (1), the result is the rather low probability of $p = 1/16$, corresponding to a chance of $1 : 15$ of hitting the centre. Consequently, it is to be expected that Helmut needs 15 ± 15 attempts until he succeeds in the sixteenth. However, anything is possible; he can be successful with his first shot or after the thirty-fourth (or not at all, if his forces or the probabilities will abandom him).

Example 2: Example 2: Archer Christina is more skilful with the bow. However, she has been practising for ten years, unlike Helmut, who has only been at it for five years. She shoots with Helmut's bow, also from a distance of $20\,m$ distance. Her shots lie with 100% probability in a circle being $d_0 = 20\,cm$ wide, which corresponds to the width of the inner blue ring in Fig.2. Let the target be again at $d_m = 8\,cm$. According to (1) the probability of hitting the centre is now 16%. Thus, Christina has a $1 : 5$ chance of being successful in hitting the target, or needs probably 6 ± 5 shots until she hits for the first time.

Example 3: Christina aims at doubling the distance between herself and the target, i.e. extend it to $40\,m$. She is certain to hit a disc of $d_0 = 40\,cm$ width with 100% probability. How big may be d_m, in order to achieve the same result as in the case of $d_0 = 20\,m$? She assumes: twice the distance requires doubling the width of the target. Correct thinking! Therefore, the target would be $d_m = 16\,cm$ wide, in order to ensure the same score as before. The extension of the newly defined target then covers the red area in Fig.2.

Conclusion: If you want to try all this out in practise, you should be aware of the requirements mentioned in the text. Above all, it must be guaranteed that all arrows (with the exception of a few outliers) land in a circle of diameter d_0, the size of certainty. On the other hand, the target d_m is chosen due to ability. In order to avoid frustration, I recommend $d_m/d_0 \approx 0.5$. This requires adjustments of distance, and inner and outer circle[3].

[3] you shoudn't miss enjoying my archery video: *Bogenschützin trifft Bogenschützen* – https://youtu.be/c4P3m4X-DHM)

Figure 2: target with a 40 cm diameter